4.9 SYSTEMS OF LINEAR DEs WITH CONSTANT COEF's

SOLVE $\frac{dx}{dt} = x - y$ BY SUBSTITUTION $\cdot x' = x - y \longrightarrow y = -x' + x$

$\frac{dy}{dt} = y - 4x$

$$y' = y - 4x$$

$$(-x'+x)' = (-x'+x) - 4x$$

$$-x'' + x' = -x' - 3x$$

$$0 = x'' - 2x' - 3x$$

$$0 = r^2 - 2r - 3$$

$$(r-3)(r+1) = 0 \rightarrow r = 3, -1$$

$$x = c_1 e^{3t} + c_2 e^{-t}$$

$$y = -x' + x$$

$$= -(c_1 e^{3t} + c_2 e^{-t})' + (c_1 e^{3t} + c_2 e^{-t})$$

$$= -3c_1 e^{3t} + c_2 e^{-t} + c_1 e^{3t} + c_2 e^{-t}$$

$$= -2c_1 e^{3t} + 2c_2 e^{-t}$$

$$x = c_1 e^{3t} + c_2 e^{-t}$$

$$y = -2c_1 e^{3t} + 2c_2 e^{-t}$$

OR

$$x = -\tfrac{1}{2}k_1 e^{3t} + \tfrac{1}{2}k_2 e^{-t}$$

$$y = k_1 e^{3t} + k_2 e^{-t}$$

DEFINE $D[y] = y'$

$D[y_1 + y_2] = (y_1 + y_2)' = y_1' + y_2' = D[y_1] + D[y_2]$

$D[cy_1] = (cy_1)' = c(y_1') = cD[y_1]$

$y'' = (y')' = D[y'] = D[D[y]] = D^2[y]$ ← NOT A POWER

SHORTHAND FOR LDO COMPOSITION

DEFINE $D^n[y] = y^{(n)}$

eg. $D^3[\sin x] = (\sin x)''' = (\cos x)'' = (-\sin x)' = -\cos x$

$(D[\sin x])^3 = ((\sin x)')^3 = \cos^3 x$

SIMILAR TO $f^{-1}(x)$ = INVERSE FUNCTION OF f

NOT $\frac{1}{f(x)}$

$(D+2)[y] = D[y] + 2[y]$
$= y' + 2y$

$c[y] = cy$
$c \in \mathbb{R}$

$3y' - 2y = (3D-2)[y]$

$4y'' + 4y' + y = (4D^2 + 4D + 1)[y]$

$L[y] = y'' + py' + qy$
$= (D^2 + pD + q)[y]$

$$L[y] = xy' = xD[y]$$

$$\begin{aligned}
L^2[y] &= L[L[y]] \\
&= L[xy'] \\
&= x(xy')' \\
&= x(y' + xy'') \\
&= x^2y'' + xy' \\
&= (x^2D^2 + xD)[y]
\end{aligned}$$

$$x^2D^2 + xD \neq (xD)^2$$

FOR THIS REASON

WE DON'T USE D NOTATION

EXCEPT FOR CONSTANT COEF'S

$$\begin{aligned}
(D+2)[(D-1)[y]] &= (D+2)[y'-y] \\
&= (y'-y)' + 2(y'-y) \\
&= y'' - y' + 2y' - 2y \\
&= y'' + y' - 2y \\
&= (D^2 + D - 2)[y]
\end{aligned}$$

COMPOSITION OF THESE D-NOTATION OPERATOR CORRESPONDING TO LDO WITH CONSTANT COEF'S IS EQUIVALENT TO MULTIPLICATION + IS COMMUTATIVE

$$(D-1)[(D+2)[y]] = (D-1)[y'+2y]$$
$$= (y'+2y)' - (y'+2y)$$
$$= y'' + 2y' - y' - 2y$$
$$= y'' + y' - 2y$$
$$= (D^2+D-2)[y]$$
$$= (D+2)[(D-1)[y]]$$

RECONSIDER $\begin{array}{l} x' = x - y \\ y' = y - 4x \end{array} \longrightarrow \begin{array}{l} x' - x + y = 0 \\ 4x + y' - y = 0 \end{array}$

$(D-1)$

$$(D-1)\,[(D-1)[x] + 1[y]] = [0]$$
$$4[x] + (D-1)[y] = 0$$
$$(D-1)^2[x] + (D-1)[y] = 0$$

SUBTRACT

$(D-1)[0]$
$= 0' - 1(0)$
$= 0$

CHAR POLY

$(r-1)^2 - 4 = 0$

$(r-1)^2 = 4$

$r - 1 = \pm 2$

$r = 1 \pm 2, = 3$ OR -1

$$\longrightarrow ((D-1)^2 - 4)[x] = 0$$
$$(D^2 - 2D - 3)[x] = 0$$
$$x'' - 2x' - 3x = 0$$
$$\vdots$$

SOLVE BY ELIMINATION

$$\boxed{x' = 3y} \qquad x' - 3y = 0 \qquad ①\ D[x] - 3[y] = 0$$

$$y' = 2x - y \longrightarrow -2x + y' + y = 0 \longrightarrow ②\ -2[x] + (D+1)[y] = 0$$

$2[①]$ $\quad 2D[x] - 6[y] = 0$

$D[②]$ $\quad -2D[x] + D(D+1)[y] = 0$

$$(D(D+1) - 6)[y] = 0$$

$$r(r+1) - 6 = 0$$

$$r^2 + r - 6 = 0$$

$$(r+3)(r-2) = 0$$

$$r = -3, 2$$

$$y = c_1 e^{-3t} + c_2 e^{2t}$$

$(D+1)[①]$ $\quad D(D+1)[x] - 3(D+1)[y] = 0$

$3[②]$ $\quad -6[x] + 3(D+1)[y] = 0$

$$(D(D+1) - 6)[x] = 0$$

$$x = k_1 e^{-3t} + k_2 e^{2t}$$

HOW ARE THE COEF'S IN X, Y RELATED?